Classical & Quantum Waves

Lecture 4-1

2) Damped Harmonic Oscillator

- All real oscillating systems are subject to damping forces. They stop oscillating if no energy is fed back into them. In applications, damping is often desired to suppress unwanted oscillations (e.g., car suspension, bridges, electronic circuits etc.)

2.1 Characteristics of the damped harmonic oscillator (DHO)

Example: Tuning fork – after strike, volume decreases, but pitch does not change!

- We infer for the ~~amplitude~~ displacement at the end of the tuning fork:

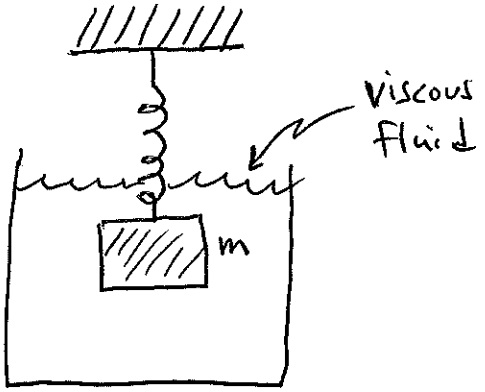
$$x(t) \simeq (\text{amplitude decreases w/ } t) \times \cos(\omega t + \phi)$$

$$\boxed{x(t) \simeq A e^{-\beta t} \cos(\omega t + \phi)}$$

decreasing amplitude

constant frequency

2.2 Equation of motion for a DHO



- Typically, in nature, we find that the faster the motion, the larger the frictional force. For a mass in a viscous fluid:

$$\boxed{F_d = -\underbrace{b}_{\text{Depends on shape of mass and viscosity of fluid}} \cdot v} \qquad [b] = N/m/s$$

$F_d \propto v$ is a good approximation for many oscillating systems.

($F_d \propto v^2$ is sometimes a good representation, e.g. "drag" from car moving in air, but this scaling is more difficult to handle mathematically.)

Eq. of motion: $\underbrace{ma = -kx}_{SHO} \underbrace{- bv}_{\text{damping term}}$

$$\rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\rightarrow \boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0}$$

w/ $\omega_0^2 = \frac{k}{m}$ natural frequency of oscillation

$\gamma = \frac{b}{m}$

↳ This eq. behaves differently for different values of γ. Need to consider multiple cases.

Case 1) $\underbrace{\text{Light damping}}$ (e.g., mass in air)

$$\frac{\gamma}{2} < \omega_0$$

· Assume $x = A_0 e^{-\beta t} \cos\omega t$

$$\frac{dx}{dt} = -A_0 e^{-\beta t}(\omega \sin\omega t + \beta \cos\omega t)$$

$$\frac{d^2x}{dt^2} = A_0 e^{-\beta t}\left[2\beta\omega \sin\omega t + (\beta^2 - \omega^2)\cos\omega t\right]$$

· Is this a solution of $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$? Insert our ansatz solution into this eq.

$$A_0 e^{-\beta t}\left[\underbrace{(2\beta\omega - \gamma\omega)}_{=0}\sin\omega t + \underbrace{(\beta^2 - \omega^2 - \gamma\beta + \omega_0^2)}_{=0}\cos\omega t\right] = 0$$

Set these terms $=0$ to find a solution

$$\rightarrow \boxed{\beta = \gamma/2}$$

$$\rightarrow \frac{\gamma^2}{4} - \omega^2 - \frac{\gamma^2}{2} + \omega_0^2 = 0$$

$$\rightarrow \boxed{\omega^2 = \omega_0^2 - \gamma^2/4}$$

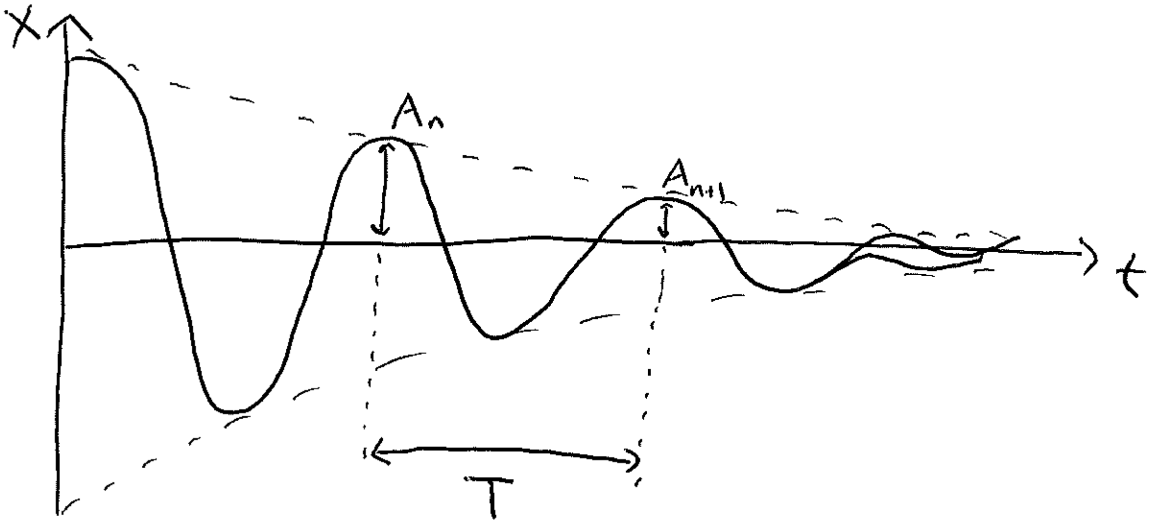
Frequency is damping dependent!

$\rightarrow$ solution: $\boxed{x = A_0 e^{-\gamma t/2}\cos(\omega t + \phi)}$ w/ $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$

$\simeq \omega_0$ (for $\frac{\gamma^2}{4} \ll \omega_0^2$)

γ, ω_0 are properties of oscillator

A_0, ϕ determined by initial conditions



$T = \frac{2\pi}{\omega}$

- Compare consecutive peaks: $\frac{A_n}{A_{n+1}} = \frac{A_0 e^{-\gamma t_0/2}}{A_0 e^{-\gamma(t_0+T)/2}} = e^{\gamma \frac{T}{2}}$

$\rightarrow \ln\left(A_n / A_{n+1}\right) = \frac{\gamma T}{2}$

"logarithmic decreasement"

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Case 2) Heavy damping ($\gamma/2 > \omega_0$)

- Damping is so large (e.g. thick oil) that system returns to EP without making an oscillation

$\rightarrow$ our previous assumption $x \sim \cos\omega t$ no longer appropriate

Try ansatz $x = e^{-\gamma t/2} f(t)$

some function

- Insert into eq. $\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$

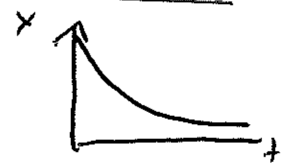
$\rightarrow \frac{d^2 f}{dt^2} + \left(\omega_0^2 - \gamma^2/4\right) f = 0 \rightsquigarrow \boxed{\frac{d^2 f}{dt^2} = \alpha^2 f}$ w/ $\alpha^2 = \frac{\gamma^2}{4} - \omega_0^2 > 0$ (heavy damping)

general solution: $f(t) = Ae^{\alpha t} + Be^{-\alpha t}$

→ Solution for heavily damped oscillator:

$$\boxed{x(t) = Ae^{\left(-\frac{\gamma}{2} + \sqrt{\gamma^2/4 - \omega_0^2}\right)t} + Be^{\left(-\frac{\gamma}{2} - \sqrt{\gamma^2/4 - \omega_0^2}\right)t}}$$

↳ exponetial decay w/o oscillation



Case 3) Critical damping (most interesting!)

Here $\frac{\gamma^2}{4} = \omega_0^2$ → $\frac{d^2f}{dt^2} = \alpha^2 f$ $\quad \alpha^2 = \frac{\gamma^2}{4} - \omega_0^2 = 0$

$$\rightarrow \frac{d^2f}{dt^2} = 0$$

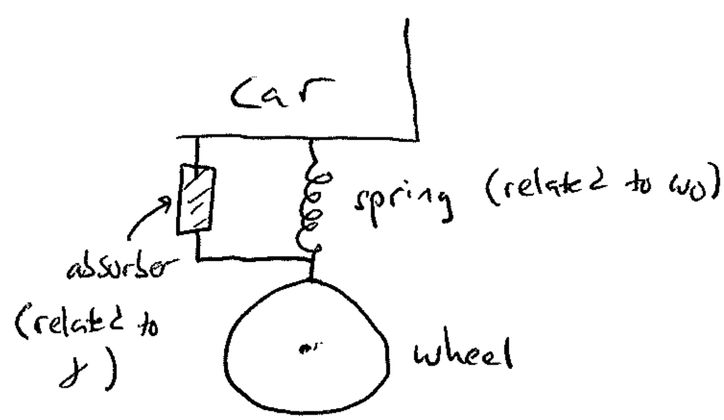
$$\rightarrow f(t) = A + Bt$$

Solution for critically damped oscillator:

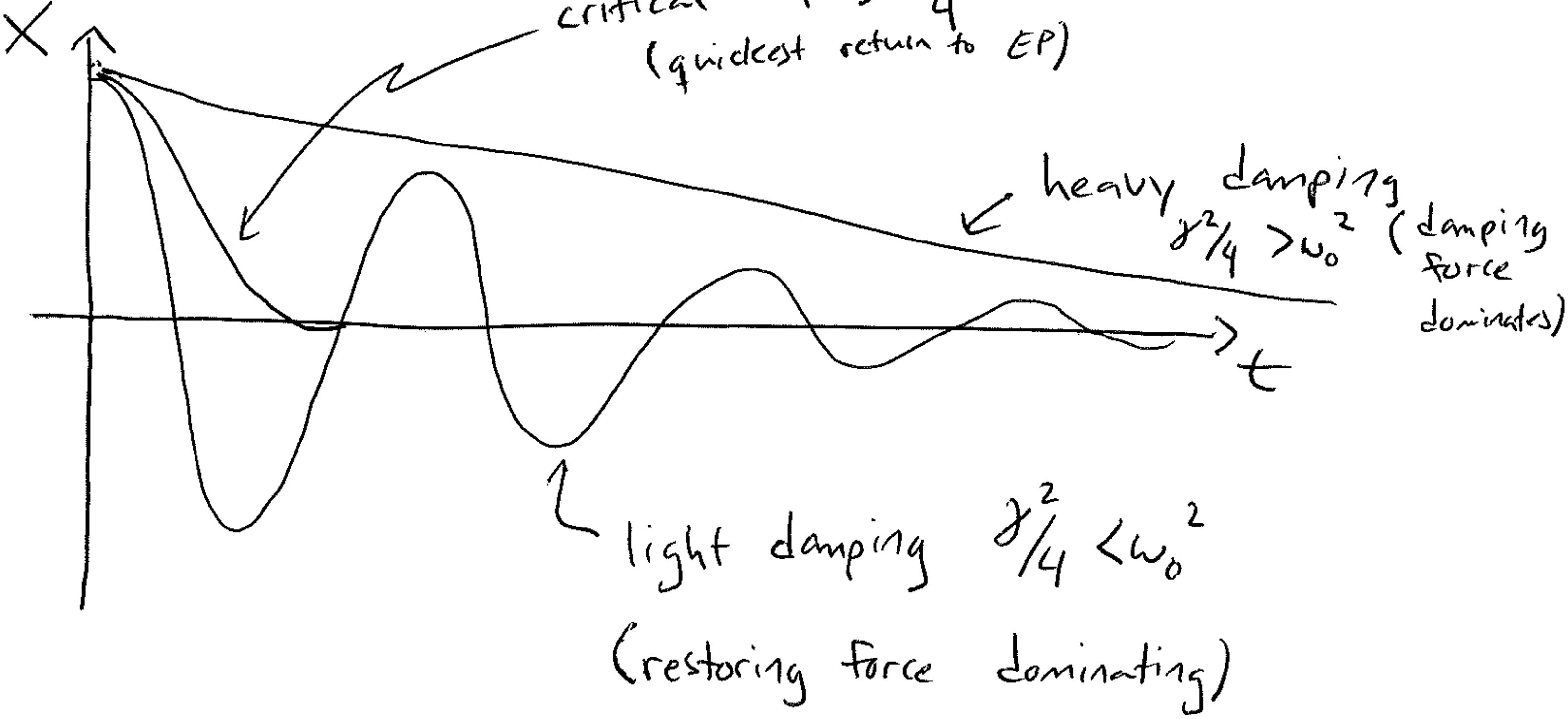
$$\boxed{x(t) = Ae^{-\gamma t/2} + Bte^{-\gamma t/2}}$$

→ system returns to E.P. in shortest possible time w/o oscillating!

Example: Shock absorber in car suspension. Spring and absorber are matched for critical damping



Summary:


X
t
critical damping $\frac{\gamma^2}{4} = \omega_0^2$
(quickest return to EP)
heavy damping $\gamma^2/4 > \omega_0^2$ (damping force dominates)
light damping $\gamma^2/4 < \omega_0^2$
(restoring force dominating)